PG – 163

Third Semester M.Sc. Degree Examination, March/April 2021
(Y2K17 Scheme) (CBCS)
MATHEMATICS
M 303 T : Functional Analysis

Time : 3 Hours Max. Marks : 70

Instructions : *1) Answer **any five** full questions.*
*2) **All** questions carry **equal** marks.*

1. a) State and prove :
   i) Cauchy-Schwartz inequality.
   ii) Minkowski inequality. **5**

   b) Show that the set Q of rational numbers is not a Banach space. **4**

   c) Let M be a closed subspace of a normed linear space N. Prove that N/M is a normed linear space. Further show that N/M is complete when N is complete. **5**

2. a) Let $(B, \|\cdot\|)$ be a Banach space such that $B = M \oplus N$, where M and N are linear subspaces of B. Then show that $B_1 = (B, \|\cdot\|_1)$ is a Banach space if M and N are closed in B, where $\|z\|_1 = \|x + y\|_1 = \|x\| + \|y\|, \forall z \in B$. **6**

   b) Show that the set B (N, N′) of all continuous linear transformations from a normed linear space N into N′ is itself a normed linear space. Further show that B(N, N′) is complete, when N′ is complete. **8**

3. a) If N is a normed linear space and $x_0 \in N$ with $x_0 \neq 0$, then show that there exists a functional $f_0 \in N^*$ such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$. **4**

   b) State and prove open mapping theorem. **10**

4. a) State and prove Banach-Steinhaus theorem. **8**

   b) Show that the mapping $T \to T^*$ is an isometric isomorphism of B(N) into $B(N^*)$ which reverses the product and preserves the identity transformation, where T is an operator on N and T^* is an operator on N^*. **4**

   c) Show that $\overline{\alpha A} = \alpha \overline{A}$, where $A \subseteq N$, where N is Banach space and α be scalar. **2**

P.T.O.

5. a) In a Hilbert space H, prove that parallelogram law and polarization identity holds. 6

b) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm. 8

6. a) If S is a non-empty subset of a Hilbert space H, then prove that $S^{\perp}$ is a closed linear subspace of H. 5

b) If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a non-zero vector z_0 in H such that $z_0 \perp M$. 5

c) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then prove that the linear subspace M + N is also closed. 4

7. a) Show that the set of all self adjoint operators in B(H), where H is a Hilbert space forms a closed linear subspace of B(H). 6

b) Show that an operator T on a Hilbert space H is self adjoint if and only if $\langle Tx, x \rangle$ is real for all x in H. 4

c) Let H be an inner product space and $S = \{e_1, e_2, ..., e_n\}$ is an orthonormal subset of H. If $x \in$ Span (S), then show that $x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$. 4

8. a) If T_1 and T_2 are normal operators on a Hilbert space H such that either commutes with the adjoint of the other, then prove that product and sum of T_1 and T_2 are also normal operators. 5

b) Define an unitary operator. Show that an unitary operator preserves the inner product and the norm. 6

c) Show that if a normal operator T has eigen value λ, then its adjoint T^* has eigen value $\bar{\lambda}$. 3
