PG – 164

III Semester M.Sc. Degree Examination, March/April 2021
(CBCS – Y2K17 Scheme)
MATHEMATICS
M304T : Linear Algebra

Time : 3 Hours | Max. Marks : 70

Instructions : *i) Answer **any five full** questions.*
*ii) **All** questions carry **equal** marks.*

1. a) If A is an algebra with unit element over a field F, then show that A is isomorphic to a sub algebra of A(V) for some vector space V over F.

   b) Define a minimal polynomial of linear transformation.

   If V is a finite dimensional vector space over a field F, then show that $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial for T is not zero.

   c) Give an example to show that $ST \neq TS$ for $S, T \in A(V)$. **(5+6+3)**

2. a) Define a range and rank of the linear transformation.

   If V is a finite dimensional vector space over a field F and $S, T \in A(V)$, then prove that $r(ST) \leq \min\{r(T), r(S)\}$.

   b) Define a characteristic roots and vectors of a linear transformation.

   Let $\lambda \in F$ be a characteristic root of $T \in A(V)$. Prove for any polynomial $q(x) \in F[x]$, $q(\lambda)$ is a characteristic root of $q(T)$.

   c) Let V be the set of all polynomials of degree $(n-1)$ or less over F. On V_{qms} let D be the transformation given by

   $(\beta_0 + \beta_1 x + \beta_2 x^2 + \ldots + \beta_{n-1}x^{n-1})D = \beta_1 + 2\beta_2 x + \ldots + (n-1)\beta_{n-1}x^{n-2}$,

   where D is the differential operator, $D = \frac{d}{dx}$. Then find the matrix of D in the following basis.

   i) $\{1, x, x^2, x^3, \ldots, x^{n-1}\}$

   ii) $\{1, 1+x, 1+x^2, \ldots, 1+x^{n-1}\}$. **(5+5+4)**

P.T.O.

3. a) Show that the product of two linear transformations is a linear transformation.

   b) Let $T : P_2(R) \to P_3(R)$ be the linear transformation defined by

   $T(a_0 + a_1x + a_2x^2) = (2a_0 + 2a_2) + (a_0 + a_1 + 3a_2)\,x + (a_1 + 2a_2)\,x^2 + (a_0 + a_2)\,x^3$.

   Then, find the matrix A of T relative to the standard basis.

   c) Define a linear functional and dual basis with examples.

   Let V be a vector space over a field F. Then prove that the double dual V^{**} is isomorphic to V. **(3+5+6)**

4. a) If $T \in A(V)$ has all its characteristic roots in F, then prove that there exists a basis of V in which the matrix of T is triangular.

   b) Define a nilpotent linear transformation. Show that two nilpotent linear transformations are similar if and only if they have the same invariants. **(6+8)**

5. a) Define a basic Jordan block and explain it with an example.

   Let $T \in A(V)$ have all its distinct characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ in F. Then show that a basis of V can be found in which the matrix T is of the form

$$\begin{pmatrix} J_1 & & & & \\ & J_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & J_k \end{pmatrix}, \text{ where } \begin{pmatrix} B_{i_1} & & & & \\ & B_{i_2} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & B_{i_r} \end{pmatrix} \text{ are basic Jordan}$$

   blocks belonging to λ_i.

   b) Let $T \in A(V)$ has a minimal polynomial $p(x) = \gamma_0 + \gamma_1 x + \ldots + \gamma_{r-1}x^{r-1} + x^r$ over F. Suppose that V is a module in a cyclic module relative to T, then prove that there exists a basis of V over F. Such that the matrix of T in this basis is of the form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\gamma_0 & -\gamma_1 & -\gamma_2 & \cdots & -\gamma_{r-1} \end{pmatrix}$$

(7+7)

6. a) Define an inner product space. State and prove the Cauchy-Schwartz inequality in an inner product space.

   b) Let $w_1 = (1, 0, 1, 0)$, $w_2 = (1, 1, 1, 1)$ and $w_3 = (0, 1, 2, 1)$ in R^4. If $\{w_1, w_2, w_3\}$ is linearly independent. Use the Gram-Schmidt orthogonalization process to compute the orthogonal vectors v_1, v_2 and v_3 and then normalize these vectors to obtain an orthonormal set.

   c) If $\{v_1, v_2 ,\ldots\ldots v_n)$ is an orthonormal set in an inner product space V and $v \in V$.

   Then prove that $\sum_{i=1}^{n} |\langle v, v_i \rangle|^2 \leq \|v\|^2$.

   Further, show that equality holds if and only if v is in the subspace spanned by $\{v_1, v_2 ,\ldots\ldots v_n)$. **(4+5+5)**

7. a) Define the following with suitable examples :

      i) Positive definite quadratic form

      ii) Positive semi definite quadratic form

      iii) Negative definite quadratic form

      iv) Negative semi definite quadratic form

      Is $Q(x) = -3x^2 + 4x^2_1 - 11x_1x_4 + 5x_2x_4 + 18x_1x_2 + 16x^2_4$ positive definite ? Justify your answer.

   b) Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$. Find the maximum value of quadratic form subject to $x^T x = 1$ and a unit vector at which this value is attained.

   c) Explain the singular value decomposition. **(6+4+4)**

8. a) Define

      i) Bilinear form

      ii) Symmetric bilinear form with an example.

   b) State and prove the Sylvester's law of inertia for real quadratic form.

   c) Find the rank and signature of the real quadratic form $x^2_1 - 4x_1x_2 + x^2_2$. **(4+8+2)**

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