PG – 142

III Semester M.Sc. Degree Examination, April/May 2022
(Y2K17/Y2K14 – CBCS)
MATHEMATICS
M 303 T : Functional Analysis

Time : 3 Hours | Max. Marks : 70

Instructions : *1) Answer **any five full** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Define a normed linear space with usual notations, show that $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ in F^n, where $1 \leq p < \infty$.
   b) Show that the set of rational numbers φ is not a Banach space. **(10+4)**

2. a) Let M be a closed linear sub space of a normed linear space N. Show that the quotient space $\frac{N}{M}$ is also normed linear space with the norm of each coset x + M defined as $\|x+M\| = \inf\{\|x+m\| : m \in M\}$. Further if N is a Banach space, then prove that $\frac{N}{M}$ is also a Banach space.
   b) If N and N′ are normed linear spaces over the same field and $T : N \rightarrow N'$ is a linear transformation, then prove that the following are equivalent.
      i) T is continuous
      ii) T is continuous at the origin
      iii) T is bounded
      iv) T(s) is bounded in N′, where $s = \{x \in N : \|x\| \leq 1\}$ is a closed unit ball in N. **(7+7)**

3. a) State and prove Hahn Banach-theorem for both real and complex cases.
   b) If N is a normed linear space and $x_0 \in N$ with $x_0 \neq 0$, then show that there exist a functional $f_0 \in N^*$ such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$. **(10+4)**

4. a) State and prove open mapping theorem.
   b) Show that the mapping $T \rightarrow T^*$ is an isometric isomorphism of B(N) into B(N*) which reverses the product and preserves the identity transformation, where T is an operator on N and T* an operator on N*.
   c) Let N be a normed linear space and X be a non empty subset of N. If f(x) is bounded for each $f \in N^*$, then show that x is bounded. **(7+4+3)**

P.T.O.

5. a) Define Hilbert space. show that the inner product and the norm in H are continuous.

   b) Prove that in a Hilbert space H, the following hold

      i) $|\langle x, y\rangle| \leq \|x\| \, \|y\|$.

      ii) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in H$.

   c) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm. **(4+4+6)**

6. a) If S is a non empty subset of Hilbert space H, then prove that $S^{\perp}$ is a closed linear subspace of H.

   b) Let H be a Hilbert space and $\{e_i\}$ be an orthonormal set in H. Then prove that the following are equivalent :

      i) $\{e_i\}_{i=1}^{n}$ is complete.

      ii) $x \perp e_i \; \forall i \Rightarrow x = 0$

      iii) $x \in H \Rightarrow x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$

      iv) $x \in H \Rightarrow \|x\|^2 = \sum_{i=1}^{n} |\langle x, e_i\rangle|^2$. **(6+8)**

7. a) Define the adjoint of an operator T on H and prove the following :

      i) $(T_1+T_2)^* = T_1^* + T_2^*$

      ii) $(\alpha T)^* = \bar{\alpha} T^*$

      iii) $\|T^*\| = \|T\|$

      iv) T is non singular $\Rightarrow$ T* is non singular.

   b) Show that an operator T on a Hilbert space H is self-adjoint if and only if (T_x, x) is real for all x in H.

   c) Show that T* is linear and continuous, where T* is adjoint of T. **(6+4+4)**

8. a) If T_1 and T_2 are normal operators on a Hilbert space H such that either commutes with the adjoint of the other, then prove that product and sum of T_1 and T_2 are also normal operators.

   b) If $P_1, P_2, \ldots, P_n$ are projections on closed linear subspaces $M_1, M_2, \ldots, M_n$ of a Hilbert space H, then prove that $P_1 + P_2 + \ldots + P_n$ is a projection iff P_i's are pair wise orthogonal.

   c) Show that if a normal operator T has eigenvalue λ, then its adjoint T* has eigenvalue $\bar{\lambda}$. **(4+7+3)**
