PG – 143

III Semester M.Sc. Degree Examination, April/May 2022
(CBCS – Y2K17/Y2K14 Scheme)
MATHEMATICS
M304T : Linear Algebra

Time : 3 Hours Max. Marks : 70

Instructions : *i) Answer **any five full** questions.*
*ii) **All** questions carry **equal** marks.*

1. a) If V is an n-dimensional vector space over F, then prove that for a given $T \in A(V)$, there exists a non trivial polynomial $q(x) \in F(x)$ of degree atmost n^2 such that $q(T) = 0$.

   b) Define minimal polynomial of a linear transformation. If V is a finite dimensional vector space over F and $T \in A_F(V)$ is invertible, then prove that T^{-1} has a polynomial expression is T over F.

   c) If V is a finite dimensional vector space over F, then prove that $T \in A_F(V)$ is regular if and only if T maps V onto itself. **(4+4+6)**

2. a) Define the rank of $T \in A(V)$. If V is a finite dimensional vector space over F, then for $S, T \in A(V)$, prove that :

   i) $r(ST) \leq r(T)$

   ii) $r(TS) \leq r(S)$

   iii) $r(TS) = r(ST) = r(T)$ for S is regular in A(V).

   b) If $\lambda \in F$ is a characteristic value of $T \in A_F(V)$, then for any $q(x) \in F(x)$, prove that $q(\lambda)$ is a characteristic root of $q(T)$.

   c) If V is an n-dimensional vector space over F and if $T \in A_F(V)$ has the matrix $m_1(T)$ in the basis $\{v_1, v_2, ..., v_n\}$ and the matrix $m_2(T)$ in the basis $\{w_1, w_2, ..., w_n\}$ of v_1, then prove that there exists a matrix C in F_n such that $m_2(T) = C\, m_1(T) C^{-1}$. **(5+4+5)**

P.T.O.

3. a) Define the composition of linear transformation. Show that the product of two linear transformations is a linear transformation.

b) Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear transformation defined by $T(a_0 + a_1x + a_2x^2) = (2a_0 + 2a_2) + (a_0 + a_1 + 3a_2)x + (a_1 + 2a_2)x^2 + (a_0 + a_2)x^3$ then find the matrix A of T relative to the standard basis.

c) Suppose that V is a finite dimensional vector space over a field F with the ordered basis $B = \{x_1, x_2, ..., x_n\}$. Let $B^* = \{f_1, f_2, ... f_n\}$ where f_i $(1 \le i \le n)$ is the i^{th} co-ordinate function with respect to B. Then show that B^* is an ordered basis of V^* and for $f \in v^*$ we have $f = \sum_{i=1}^{n} f(x_i)f_i$. **(5+4+5)**

4. a) If $W \subset V$ is an invariant subspace under T, then prove that T induces a linear transformation $\overline{T}$ on $\overline{V}$. If T satisfies a polynomial $q(x) \in F[x]$, then prove that $\overline{T}$ also satisfies q(x). Further if $p_1(x)$ is the minimal polynomial for $\overline{T}$ over F and p(x) is that for T, then prove that $p_1(x)$ divides p(x).

b) Define triangular canonical form. If $T \in A_F(V)$ has all its characteristic roots in F, then show that there exists a basis of V in which the matrix of T is triangular. **(7+7)**

5. a) Define a nilpotent linear transformation. Show that two nilpotent transformation are similar if and only if they have the same invariants.

b) If $T \in A_F(V)$ has minimal polynomial $p(x) = q_1(x)^{l_1} q_2(x)^{l_2} ... q_k(x)^{l_k}$ over F, where $q_1(x), q_2(x), ... q_k(x)$ are irreducible distinct polynomials in F[x], then prove that there exists a basis of V in which the matrix of T is of the form

$$\begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & R_n \end{bmatrix} \text{ where each } R_i = \begin{bmatrix} C\left(q_1(x)^{e_{i_1}}\right) & & \\ & \ddots & \\ & & C(q_i(x)^{e_{i_n}}) \end{bmatrix}$$

where $e_{i_1} \ge e_{i_2} \ge ... \ge e_{i_n}$. **(7+7)**

6. a) Let u and v be two vectors in an inner product space v such that $\|u+v\| = \|u\| + \|v\|$. Prove that u and v are linear dependent vectors. Give an example to show that the converse of this statement is not true.

   b) Define an orthogonal compliment. Let u = (–1, 4, –3) be a vector in the inner product space with standard inner product. Find a basis of the subspace $u^{\perp}$ of $\mathbb{R}^3$.

   c) State and prove Bessel's inequality. **(5+4+5)**

7. a) Define a quadratic form. Explain the classification of quadratic form with suitable example.

   b) Is $Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$ positive definite ? Explain.

   c) Decompose the following matrix into its singular value decomposition.

   $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$. **(5+4+5)**

8. a) Define :

   i) Bilinear form.

   ii) Symmetric bilinear form with an example.

   Let $\mathbb{B}$ be a bilinear form on a finite dimensional vector space V and let β be an ordered basis of V. Then show that $\mathbb{B}$ is symmetric if and only if $\Psi_\beta(\mathbb{B})$ is symmetric.

   b) Show that two real symmetric matrices are congruent if and only if they have same rank and signature.

   c) Find the rank and signature of the real quadratic form $x_1^2 - 4x_1x_2 + x_2^2$. **(6+6+2)**
