I Semester M.Sc. Degree Examination, August/September 2021
(CBCS-Y2K17/Y2K14)
MATHEMATICS
M 101 T : Algebra – I

Time : 3 Hours Max. Marks : 70

***Instructions** : 1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Let $\phi : G \to \overline{G}$ be a homomorphism with Kernel K and let $\overline{N}$ be a normal subgroup of $\overline{G}$ and $N = \{g \in G : \phi(g) \in \overline{N}\}$. Prove that $G/N \cong \overline{G}/\overline{N}$.
   b) Write Aut (K_4), where K_4 is Klein-four group. Hence illustrate that the automorphism group of an abelian group need not be abelian.
   c) State and prove Cayley's theorem. **(4+5+5)**

2. a) State and prove the Orbit-Stabilizer theorem.
   b) If G is a finite group and $a \in G$, prove that $C_a = \frac{O(G)}{O(N(a))}$, where N(a) is the normalizer of 'a' and C_a is the conjugacy class of a in G.
   c) Prove that every group of order p^2, for some prime 'p' is abelian. **(5+4+5)**

3. a) State and prove Sylow's first theorem.
   b) Let G be a group of order pq, where p and q are primes with $p < q$ and $q \equiv 1 \pmod{p}$. Show that G is non-abelian. **(8+6)**

4. a) Define a simple group. Show that a group of order 28 is solvable but not simple.
   b) If a group G has a composition series, then show that all its composition series are pairwise equivalent.
   c) Give an example of a non abelian solvable group. **(5+6+3)**

5. a) Let R be a commutative ring with unity whose ideals are (O) and R only. Prove that R is a field.
   b) If U is an ideal of a ring R and $[R : U] = \{x \in R; rx \in U, \forall r \in R\}$, then prove that [R : U] is an ideal of R containing U.
   c) State and prove fundamental theorem of homomorphism for rings. **(5+4+5)**

P.T.O.

6. a) Define principal ideal of a ring R. Show that the ring z of all integers is a principal ideal ring.
   b) Prove that the ideal of the ring Z of integers is maximal if and only if it is generated by some prime integer in Z.
   c) Show that any two isomorphic integral domains have isomorphic quotient fields. **(5+5+4)**

7. a) Define a Euclidean ring. Let x = a + ib and y = c + id be any two elements in z[i] – {0}. Prove that it is an Euclidean ring.
   b) Let R be an Euclidean ring and a, b $\in$ R be non-zero with 'b' non-unit. Then prove that d(a) < d(ab).
   c) State and prove the unique factorization theorem. **(5+3+6)**

8. a) Prove that deg (fg) = deg (f) + deg (g), for f, g $\in$ R[x]. Further if R is an integral domain, then show that R[x] is also an integral domain.
   b) State and prove Euclid's algorithm for polynomials over a field.
   c) Let A = $(x^2 + x + 1)$ be an ideal generated by $x^2 + x + 1 \in z_2[x]$. Verify that A is a maximal ideal in $z_2[x]$. **(4+6+4)**

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