PG – 647

First Semester M.Sc. Degree Examination, August/September 2021
(CBCS – Y2K17/Y2K14)
MATHEMATICS
M 102 T : Real Analysis

Time : 3 Hours Max. Marks : 70

Instructions : *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Evaluate $\int_0^3 x\, d\{[x]\}$ where [x] is the maximum integer function. 4

   b) If $f \in R[\alpha]$ on [a, b], then prove that $\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha = \int_a^b f d\alpha = \lambda[\alpha(b) - \alpha(a)]$, where $\lambda \in [m, M]$. 5

   c) If P* is a refinement of partition P of [a, b], then show that $L(P, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P, f, \alpha)$. 5

2. a) If $f \in R[\alpha]$ on [a, b], $f(x) \in [m, M]$ for all $x \in [a, b]$ and ϕ is continuous on [m, M], then prove that $\phi \bullet f \in R[\alpha]$ on [a, b]. 7

   b) If f(x) is continuous on [a, b] and $\alpha(x)$ be monotonic on [a, b], prove that $\int_a^b f d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \alpha(\xi)[f(b) - f(a)]$ where $\xi \in (a, b)$. 5

   c) Give an example of a function f such that $|f| \in R[\alpha]$ on [0, 1] and $f \notin R[\alpha]$ on [0, 1]. 2

3. a) Consider the functions $\beta_1(x)$ and $\beta_2(x)$ defined as follows :

$$\beta_1(x) = \begin{cases} 0, & \text{when } x \leq 0 \\ 1, & \text{when } x > 0 \end{cases}$$

$$\beta_2(x) = \begin{cases} 0, & \text{when } x < 0 \\ 1, & \text{when } x \geq 0 \end{cases}$$

   Verify whether $\beta_1(x) \in R[\beta_2(x)]$ on [– 1, 1]. 7

   b) If f and ϕ are continuous on [a, b] and ϕ is strictly increasing on [a, b] and ψ is an inverse function of ϕ, then prove that $\int_a^b f(x)dx = \int_{\phi(a)}^{\phi(b)} f(\psi(g))\, d\psi(g)$. 5

   c) Prove that a function of bounded variation on [a, b] is bounded. 2

P.T.O.

4. a) State and prove Cauchy's principle for uniform convergence of

i) $\{f_n(x)\}$ on [a, b]

ii) $\sum_{n=1}^{\infty} f_n(x)$ on [a, b] **10**

b) Show that for $-1 < x < 1$, the series $\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \ldots = \frac{1}{1-x}$. **4**

5. a) Let $\{f_n(x)\}$ be a sequence of differentiable functions such that the sequence converges for atleast one-point $t \in [a, b]$. If the sequence of the derivatives of $f_n(x)$, that is $\{f_n'(x)\}$ is uniformly convergent to F(x) on [a, b], then prove that $\{f_n(x)\}$ is uniformly convergent to f(x) on [a, b] and that $f'(x) = F(x), \forall x \in [a, b]$. **7**

b) Let $\{f_n(x)\}$ be a sequence of functions uniformly convergent to f(x) on [a, b] and each $f_n(x) \in R[a, b]$. Prove the following :

i) $f(x) \in R[a, b]$,

ii) $\int_a^x \lim_{n\to\infty} f_n(t)dt = \lim_{n-\infty} \int_a^x f_n(t)dt$. **7**

6. a) State and prove the Hiene-Borel theorem. **7**

b) Define a K-cell. Prove that every K-cell is compact. **7**

7. a) Let 'E' be an open subset of R^n and $f : E \to R^n$ be differentiable at a point $x_0 \in E$. Let 'F' be an open subset of R^n containing 'E' and $g : F \to R^k$ be differentiable at $f(x_0)$. If $\phi = g \circ f : E \to R^k$, then prove that ϕ is differentiable at $x_0 \in E$ and $\phi'(x_0) = g'(f(x_0)) \circ f'(x_0)$. **6**

b) Let $T : R^n \to R^m$ be a mapping with $T = (T_1, T_2, \ldots, T_m)$. Prove that 'T' is linear transformation if and only if T_i (i = 1, 2,...,m) are linear transformations. **2**

c) If $\phi : X \to X$ is a contraction on a complete metric space X, then prove that ϕ has a unique fixed point. **6**

8. State and prove the inverse function theorem. **14**
